

## GROSS-HOPKINS DUALITY

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In [8] Hopkins and Gross state a theorem revealing a profound relationship between two different kinds of duality in stable homotopy theory. A proof of a related but weaker result is given in [3], and we understand that Sadofsky is preparing a proof that works in general. Here we present a proof that seems rather different and complementary to Sadofsky's. We thank I-Chiau Huang for help with Proposition 18, and John Greenlees for helpful discussions.

We first indicate the context of the Hopkins-Gross theorem. Cohomological duality theorems have been studied in a number of contexts; they typically say that

$$H^k(X^*) = H^{d-k}(X)^\vee$$

for some class of objects  $X$  with some notion of duality  $X \leftrightarrow X^*$  and some type of cohomology groups  $H^k(X)$  with some notion of duality  $A \leftrightarrow A^\vee$  and some integer  $d$ . For example, if  $M$  is a compact smooth oriented manifold of dimension  $d$  we have a Poincaré duality isomorphism

$$H^k(M; \mathbb{Q}) = \text{Hom}(H^{d-k}(M; \mathbb{Q}), \mathbb{Q})$$

(so here we just have  $M^* = M$ ). For another example, let  $S$  be a smooth complex projective variety of dimension  $d$ , and let  $\Omega^d$  be the sheaf of top-dimensional differential forms. Then for any coherent sheaf  $F$  on  $S$  we have a Serre duality isomorphism

$$H^k(S; \text{Hom}(F, \Omega^d)) = \text{Hom}(H^{d-k}(S; F), \mathbb{C}).$$

This can be seen as a special case of the Grothendieck duality theorem for a proper morphism [7], which is formulated in terms of functors between derived categories. There is a well-known analogy between Boardman's stable homotopy category of spectra and the derived category of sheaves over a scheme, and, inspired by this, Neeman has used tools from homotopy theory to prove the main facts about Grothendieck duality [20]. This leads one to expect that there should be a kind of duality theorem in the stable homotopy category itself. However, experience suggests that such theorems require finiteness conditions which are not satisfied in the category of spectra. The subcategory of  $K(n)$ -local spectra [21, 12] has much better finiteness properties and thus seems a better place to look for duality phenomena. The Hopkins-Gross theorem is a kind of analog of Serre duality in the  $K(n)$ -local stable homotopy category.

We next explain the result in question in more detail; our formulation will be compared with various other possible formulations in Remark 21. Fix a prime  $p$  and an integer  $n > 0$  and let  $L_n$  be the Bousfield localisation functor [21] with respect to  $E(n)$ . Let  $MS$  be the fibre of the natural map  $L_n S \rightarrow L_{n-1} S$ , and let  $\hat{I} = IMS$  be its Brown-Comenetz dual, which is characterised by the existence of a natural isomorphism

$$[X, \hat{I}] = \text{Hom}(\pi_0(MX), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\pi_0(MS \wedge X), \mathbb{Q}/\mathbb{Z}).$$

This lies in the  $K(n)$ -local category  $\mathcal{K}$ , which is a symmetric monoidal category with unit  $\hat{S} = L_{K(n)} S$  and smash product  $X \hat{\wedge} Y = L_{K(n)}(X \wedge Y)$ . The formal properties of the category  $\mathcal{K}$  are studied in detail in [12]. If  $X \in \mathcal{K}$  we write  $\hat{I}X = F(X, \hat{I})$  and  $DX = F(X, \hat{S})$ ; these are the natural analogues of the Brown-Comenetz dual and the Spanier-Whitehead dual of  $X$  in the  $K(n)$ -local context. The following result is contained in [8, Theorem 6]; it is convenient for us to state it separately.

**Proposition 1** (Hopkins-Gross). *The spectrum  $\hat{I}$  is invertible: there exists a spectrum  $\hat{I}^{-1} \in \mathcal{K}$  with  $\hat{I} \hat{\wedge} \hat{I}^{-1} = \hat{S}$ . It follows that  $\hat{I}X = F(X, \hat{I}) = \hat{I} \hat{\wedge} DX$  for all  $X \in \mathcal{K}$ .*

*Proof.* The original Hopkins-Gross proof has never appeared, but a proof of the first claim was given in [12, Theorem 10.2(e)]. The second follows purely formally from the first: the functor  $\widehat{I}\widehat{\wedge}(-): \mathcal{K} \rightarrow \mathcal{K}$  is an equivalence, so we have

$$[Y, \widehat{I}X] = [X \widehat{\wedge} Y, \widehat{I}] = [\widehat{I}^{-1} \widehat{\wedge} X \widehat{\wedge} Y, S] = [\widehat{I}^{-1} \widehat{\wedge} Y, DX] = [Y, \widehat{I} \widehat{\wedge} DX].$$

Yoneda's lemma completes the argument.  $\square$

Our next problem is to study these dualities using a suitable cohomology theory, which we call Morava  $E$ -theory. It is a 2-periodic Landweber exact theory with coefficient ring

$$E_* = W[u_1, \dots, u_{n-1}][u, u^{-1}]$$

where  $|u_k| = 0$  and  $|u| = 2$  and  $W$  is the Witt ring of the finite field  $\mathbb{F}_q = \mathbb{F}_{p^n}$ . We make this a  $BP_*$ -algebra by the map sending  $v_k$  to  $u^{p^k-1}u_k$ , where  $u_0 = p$  and  $u_n = 1$  and  $u_k = 0$  for  $k > n$ . This gives rise to a Landweber exact formal group law and thus a multiplicative cohomology theory. We write  $E$  for the representing spectrum, which is  $K(n)$ -local, and is a wedge of finitely many suspended copies of the spectrum  $\widehat{E}(n)$ . If  $X \in \mathcal{K}$  then we put  $E_t^\vee X = \pi_t(E \widehat{\wedge} X)$ ; this is the natural version of  $E$ -homology to use in a  $K(n)$ -local context. Our goal is to describe  $E_*^\vee \widehat{I}$ . The proposition above is actually equivalent (by [9, Theorem 1.3]) to the statement that  $E_*^\vee \widehat{I} \simeq E_*$  as  $E_*$ -modules (up to suspension). However, this is not sufficient for the applications; we need pin down the action of the group  $\Gamma$  of multiplicative automorphisms of  $E$  (a version of the Morava stabiliser group) as well as the  $E_*$ -module structure. This provides the input for various Adams-type spectral sequences. To explain the answer, we recall that there is a canonical map  $\det: \Gamma \rightarrow \mathbb{Z}_p^\times$  called the reduced determinant. If  $M$  is an  $E_0$ -module with compatible action of  $\Gamma$  (in short, an  $E^0$ - $\Gamma$ -module) then we write  $M[\det]$  for the same  $E^0$ -module with the  $\Gamma$ -action twisted by  $\det$ .

**Theorem 2** (Hopkins-Gross). *There is a natural  $\Gamma$ -equivariant isomorphism*

$$E_t^\vee \widehat{I} = E_{n-n^2+t}[\det].$$

The proof of this result falls naturally into two parts. One part is the algebraic analysis of equivariant vector bundles on the Lubin-Tate deformation space, which is outlined by Hopkins and Gross in [8], with full details provided by the same authors in [6]. The other part is a topological argument to make contact with Morava  $E$ -theory. The theorem as stated implies a description of  $E_t^\vee \widehat{I}(Z)$  when  $Z$  is a finite complex of type  $n$ , and Devinatz [3] has given a proof (following a sketch by Hopkins) that this description is valid when  $p > (n^2 + n + 2)/2$ .

The rest of the paper will constitute our proof of the full theorem.

We first need some information about the structure of the group  $\Gamma$ . We will assume that the reader is familiar with the general idea of the relationship between division algebras, automorphisms of formal groups and cohomology operations. We therefore give just enough detail to pin down our group  $\Gamma$  among the various possible versions of the Morava stabiliser group. The original reference for most of these ideas is [19], and [2] is a good source for many technical points.

**Definition 3.** Let  $\phi$  be the unique automorphism of  $W$  such that  $\phi(a) = a^p \pmod{p}$  for all  $a \in W$ ; note that  $\phi^n = 1$ . Let  $D$  be the noncommutative ring obtained from  $W$  by adjoining an element  $s$  satisfying  $sa = \phi(a)s$  (for  $a \in W$ ) and  $s^n = p$ ; thus  $\mathbb{Q} \otimes D$  is the central division algebra over  $\mathbb{Q}_p$  of rank  $n^2$  and invariant  $1/n$ . There is a unique way to extend  $\phi$  to an automorphism of  $D$  with  $\phi(s) = s$ , and we still have  $\phi^n = 1$ . This gives an action of the cyclic group  $C = \langle \phi | \phi^n = 1 \rangle$  on  $D$  and thus allows us to form the semidirect product  $D^\times \ltimes C$ .

**Proposition 4.** *There is a natural isomorphism  $\Gamma \simeq D^\times \ltimes C$ .*

*Proof.* Write  $\Gamma' = D^\times \ltimes C$ . Let  $F_0$  be the unique  $p$ -typical formal group law over  $\mathbb{F}_q$  with  $[p](x) = x^q$ , and let  $\mathbb{G}_0$  be the formal group scheme over  $S_0 := \text{spec}(\mathbb{F}_q)$  associated to  $F_0$ . If  $\omega \in \mathbb{F}_q = W/p$  then there is a unique lift  $\hat{\omega} \in W$  such that  $\hat{\omega}^q = \hat{\omega}$ , and there is also an endomorphism  $\mu(\omega)$  of  $\mathbb{G}_0$  given by  $x \mapsto \omega x$ . We also have an endomorphism  $\sigma$  of  $\mathbb{G}_0$  given by  $x \mapsto x^p$ . It is well-known that there is a unique ring map  $\theta: D \rightarrow \text{End}(\mathbb{G}_0)$  such that  $\theta(s) = \sigma$  and  $\theta(\hat{\omega}) = \mu(\omega)$  for all  $\omega \in \mathbb{F}_q$ , and moreover that  $\theta$  is an isomorphism. On the other hand, we can let  $C$  act on  $S_0$  via the Frobenius automorphism  $\phi: S_0 \rightarrow S_0$ . As the coefficients

of  $F_0$  lie in  $\mathbb{F}_p$ , there is a natural identification  $(\phi^k)^*\mathbb{G}_0 = \mathbb{G}_0$ . Using this, it is easy to identify  $\Gamma'$  with the group of pairs  $(\alpha_0, \beta_0)$ , where  $\beta_0: S_0 \rightarrow S_0$  is an isomorphism of schemes and  $\alpha_0: \mathbb{G}_0 \rightarrow \beta_0^*\mathbb{G}_0$  is an isomorphism of formal groups over  $S_0$ .

Now put  $S = \mathrm{spf}(E^0)$  and  $\mathbb{G} = \mathrm{spf}(E^0\mathbb{C}P^\infty)$  so that  $\mathbb{G}$  is a formal group over  $S$  and is the universal deformation of  $\mathbb{G}_0$  in the sense of Lubin and Tate [17] (see also [24, Section 6] for an account in the present language.) Let  $\Gamma''$  denote the group of pairs  $(\alpha, \beta)$  where  $\beta: S \rightarrow S$  and  $\alpha: \mathbb{G} \rightarrow \beta^*\mathbb{G}$ . As  $S_0$  is the subscheme of  $S$  defined by the unique maximal ideal in  $E^0 = \mathcal{O}_S$  we see that  $\beta(S_0) = S_0$ . As  $\mathbb{G}_0 = S_0 \times_S \mathbb{G}$ , we also see that  $\alpha(\mathbb{G}_0) = \mathbb{G}_0$ , so we get a homomorphism  $\Gamma'' \rightarrow \Gamma'$  sending  $(\alpha, \beta)$  to  $(\alpha|_{S_0}, \beta|_{\mathbb{G}_0})$ ; deformation theory tells us that this is an isomorphism. The general theory of Landweber-exact ring spectra [14] gives an isomorphism  $\Gamma'' \rightarrow \mathrm{Aut}(E) = \Gamma$ ; see [24, Section 8.7] for an account in the present language.  $\square$

For the next proposition, we note that the topological ring  $\mathbb{Z}_p[[\Gamma]]$  has both a left and a right action of  $\Gamma$ , and these actions are continuous and commute with each other. We can use the left action to define continuous cohomology groups  $H^*(\Gamma; \mathbb{Z}_p[[\Gamma]])$ , and the right action gives an action of  $\Gamma$  on these cohomology groups.

**Proposition 5.** *We have  $H^{n^2}(\Gamma; \mathbb{Z}_p[[\Gamma]]) = \mathbb{Z}_p$ , and the natural action of  $\Gamma$  on this module is trivial. Moreover, we have  $H^k(\Gamma; \mathbb{Z}_p[[\Gamma]]) = 0$  for  $k \neq n^2$ .*

*Proof.* First, we note that  $\Gamma$  is a  $p$ -adic analytic group of dimension  $n^2$  over  $\mathbb{Z}_p$ . Duality phenomena in the cohomology of profinite groups have been studied for a long time [23, 15], but the more recent paper [25] is the most convenient reference for the particular points that we need. Write  $U = 1 + pD < D^\times < \Gamma$ , which is a torsion-free open subgroup of finite index in  $\Gamma$ . It follows from [25, Corollary 5.1.6] that  $U$  is a Poincaré duality group of dimension  $n^2$  in the sense used in that paper, which means precisely that  $U$  has cohomological dimension  $n^2$  and  $H^{n^2}(U; \mathbb{Z}_p[[U]]) = \mathbb{Z}_p$  and the other cohomology groups are trivial. As  $U$  has finite index in  $\Gamma$ , Shapiro's lemma gives an isomorphism  $H^*(\Gamma; \mathbb{Z}_p[[\Gamma]]) = H^*(U; \mathbb{Z}_p[[U]])$ ; this proves the proposition except for the fact that  $\Gamma$  acts trivially. There is a unique way to let  $\Gamma$  act on  $D$  such that the subgroup  $D^\times$  acts by conjugation and the subgroup  $C$  acts via  $\phi$ . On the other hand,  $\Gamma$  acts on  $U$  by conjugation and thus on the  $\mathbb{Q}_p$ -Lie algebra  $L$  of  $U$ . There is an evident  $\Gamma$ -equivariant isomorphism  $L = \mathbb{Q} \otimes D$ . Using the results of [25, Section 5] we get a  $\Gamma$ -equivariant isomorphism

$$\mathbb{Q} \otimes H^{n^2}(\Gamma; \mathbb{Z}_p[[\Gamma]]) = \mathbb{Q} \otimes H^{n^2}(U; \mathbb{Z}_p[[U]]) = \Lambda^{n^2} L = \mathbb{Q} \otimes \Lambda^{n^2} D.$$

(This is just a tiny extension of an argument of Lazard, which could be applied directly if  $\Gamma$  were torsion-free.) We write  $\mathrm{ad}(\gamma)$  for the determinant of the action of  $\gamma \in \Gamma$  on  $D$ ; it is now enough to check that  $\mathrm{ad} = 1$ .

Suppose  $a \in D^\times$ , so  $a$  acts on  $D$  by  $x \mapsto axa^{-1}$ . Let  $K$  be the subfield of  $\mathbb{Q} \otimes D$  generated over  $\mathbb{Q}_p$  by  $a$ . Put  $d = \dim_{\mathbb{Q}_p} K$ , so that  $\mathbb{Q} \otimes D \simeq K^{n^2/d}$  as left  $K$ -modules. Using this, we see that the determinant of left multiplication by  $a$  is just  $N_{K/\mathbb{Q}_p}(a)^{n^2/d}$ . By a similar argument, the determinant of right multiplication by  $a^{-1}$  is  $N_{K/\mathbb{Q}_p}(a)^{-n^2/d}$ . It follows that the conjugation map has determinant one, so that  $\mathrm{ad}(D^\times) = 1$ . Moreover, the action of  $\phi \in C$  on  $\mathbb{Q} \otimes D$  is the same as the action of  $s \in \mathbb{Q} \otimes D$  by conjugation, which has determinant one by the same argument.  $\square$

We next discuss some useful generalities about  $E$ -module spectra. As in [12, Appendix A], we say that an  $E^0$ -module  $M$  is *pro-free* if it is the completion at  $I_n$  of a free module.

**Lemma 6.** *Let  $M$  be a  $K(n)$ -local  $E$ -module spectrum. Then the following are equivalent:*

- (a)  $\pi_1 M = 0$  and  $\pi_0 M$  is pro-free as an  $E^0$ -module.
- (b)  $M$  is a coproduct in  $\mathcal{K}$  of copies of  $E$ .
- (c)  $M$  is a retract of a product of copies of  $E$ .

Moreover, if  $\mathcal{M}$  is the category of  $E$ -modules for which these conditions hold and  $\mathcal{M}_0$  is the category of pro-free  $E$ -modules then the functor  $\pi_0: \mathcal{M} \rightarrow \mathcal{M}_0$  is an equivalence.

*Proof.* Suppose (b) holds, so  $M = \bigvee_\alpha E$  say. Here the coproduct is the  $K(n)$ -localisation of the ordinary wedge, which means that  $M = \varprojlim_J \bigvee_\alpha E/J$ , where  $J$  runs over a suitable family of open ideals in  $E^0$ . In this second expression, it makes no difference whether the wedge is taken in  $\mathcal{K}$  or in the category of all spectra,

so  $\pi_* \bigvee_{\alpha} E/J = \bigoplus_{\alpha} E_*/J$ , and it follows easily that  $\pi_0 M$  is the completion of  $\bigoplus_{\alpha} E_0$  and  $\pi_1 M = 0$ . Thus (b)  $\Rightarrow$  (a).

Conversely, if (a) holds, choose a topological basis  $\{e_{\alpha}\}$  for  $\pi_0 M$  and use it to construct a map  $f: \bigvee_{\alpha} E \rightarrow M$  of  $E$ -modules in the usual way. Then  $\pi_0 \bigvee_{\alpha} E$  is the completion of  $\bigoplus_{\alpha} E_0$  and so  $\pi_*(f)$  is an isomorphism, so (b) holds. Using this we find that for any  $K(n)$ -local  $E$ -module  $N$ , the group of  $E$ -module maps  $M \rightarrow N$  is just  $\prod_{\alpha} \pi_0 N = \text{Hom}_{E_0}(\pi_0 M, \pi_0 N)$ . If we write  $\mathcal{M}$  for the category of  $E$ -modules satisfying (a) and (b), it is now clear that  $\pi_0: \mathcal{M} \rightarrow \mathcal{M}_0$  is an equivalence.

We know from [12, Appendix A] that  $\mathcal{M}_0$  is closed under arbitrary products and retracts, and that any pro-free  $E^0$ -module is a retract of a product of copies of  $E^0$ . Given this, we can easily deduce that (a)  $\Leftrightarrow$  (c).  $\square$

**Definition 7.** We say that an  $E$ -module spectrum  $M$  is *pro-free* if it satisfies the conditions of the lemma.

**Corollary 8.** If  $M$  is pro-free and  $\{X_{\alpha}\}$  is the diagram of small spectra over  $X$  then  $[X, M] = \varprojlim_{\alpha} [X_{\alpha}, M]$ .

*Proof.* If  $M = E$  then this holds by a well-known compactness argument based on the fact that  $E^0 Y$  is finite for all small  $Y$ . As any  $M$  can be written as a retract of a product of copies of  $E$ , the claim follows in general.  $\square$

**Lemma 9.** Let  $M$  and  $N$  be pro-free  $E$ -modules, and use the  $E$ -module structure on  $M$  to make  $M \hat{\wedge} N$  an  $E$ -module. Then  $M \hat{\wedge} N$  is pro-free.

*Proof.* It is easy to reduce to the case  $M = N = E$ . By the argument of [12, Proposition 8.4(f)] it suffices to check that  $(E/I_n)_* E$  is concentrated in even degrees. We know by standard calculations that  $(E/I_n)_* BP = E_*[t_k \mid k > 0]/I_n$  (with  $|t_k| = 2(p^k - 1)$ ) and that  $(E/I_n)_* E = (E/I_n)_* BP \otimes_{BP_*} E_*$  by Landweber exactness, and the claim follows.  $\square$

**Lemma 10.** Let  $C(\Gamma, E_0)$  be the ring of continuous functions from  $\Gamma$  (with its profinite topology) to  $E_0$  (with its  $I_n$ -adic topology). Then  $C(\Gamma, E_0)$  is pro-free.

*Proof.* Write  $\widehat{\mathbb{F}}_q = \{a \in W \mid a^q = a\}$ ; the reduction map  $\widehat{\mathbb{F}}_q \rightarrow \mathbb{F}_q$  is well-known to be an isomorphism. For any  $d \in D^{\times}$  there is a unique way to write  $d = \sum_{k \geq 0} \tau_k(d) s^k$  with  $\tau_k(d) \in \widehat{\mathbb{F}}_q$  and  $\tau_0(d) \neq 0$ . This defines functions  $\tau_k: D^{\times} \rightarrow E_0$ , and  $\tau_k$  is constant on the cosets of  $1 + s^{k+1}D$  and thus is locally constant. Lagrange interpolation shows that the evident map from  $\mathbb{F}_q[\tau]/(\tau^q - \tau)$  to the ring of all functions  $\mathbb{F}_q \rightarrow \mathbb{F}_q$  is surjective, and thus an isomorphism by dimension count. Our maps  $\tau_i$  give a bijection

$$(\tau_0, \dots, \tau_k): D^{\times}/(1 + s^{k+1}D) \rightarrow \mathbb{F}_q^{\times} \times \mathbb{F}_q^k.$$

Putting these facts together, we see that the ring of functions from  $D^{\times}/(1 + s^{k+1}D)$  to  $E_0$  is generated over  $E_0$  by the functions  $\tau_0, \dots, \tau_k$  subject only to the relations  $\tau_i^q = \tau_i$  and  $\tau_0^{q-1} = 1$ . The direct limit of these rings as  $k$  tends to  $\infty$  is the ring of all locally constant functions from  $\Gamma$  to  $E_0$ , which is thus isomorphic to

$$E_0[\tau_k \mid k \geq 0]/(\tau_0^{q-1} - 1, \tau_k^q - \tau_k).$$

Now let  $e_i: C \rightarrow E_0$  be the characteristic function of  $\{\phi^i\}$  (for  $i = 0, \dots, n-1$ ), so the ring of functions from  $C$  to  $E_0$  is just  $E[e_0, \dots, e_{n-1}]/(e_i e_j - \delta_{ij} e_i)$ .

Recall that  $\Gamma = D^{\times} \ltimes C$ ; this can be identified with  $D^{\times} \times C$  as a set, so the ring of locally constant functions from  $\Gamma$  to  $E_0$  is just the tensor product of the rings for  $D^{\times}$  and  $C$ , which we now see is a free  $E_0$ -module. The ring of all continuous functions is the completion of the ring of locally constant functions, and thus is pro-free.  $\square$

We next want to identify  $E_0^{\vee} E$  with  $C(\Gamma, E_0)$ . This is in some sense well-known, but the details are difficult to extract from the literature in a convenient form, so at the suggestion of the referee we indicate a proof.

It will first be helpful to have a more coherent view of the topologies on our various algebraic objects. Recall that a  $K(n)$ -local spectrum  $W$  is *small* if it is a retract of the  $K(n)$ -localisation of a finite spectrum of type  $n$ . For any  $X, Y \in \mathcal{K}$  we define a topology on  $[X, Y]$  whose basic neighbourhoods of 0 are the kernels

of maps  $u^*: [X, Y] \rightarrow [W, Y]$  such that  $W$  is small and  $u: W \rightarrow X$ . We call this the “natural topology”; its formal properties are developed in [12, Section 11]. It is shown there that the natural topology on  $E_0$  is the same as the  $I_n$ -adic topology, which is easily seen to be the same as the profinite topology.

**Lemma 11.** *The natural topology on  $\Gamma \subset [E, E]$  is also the same as the profinite topology.*

*Proof.* Consider a map  $u: W \rightarrow E$  and the resulting map  $u^*: \Gamma \rightarrow E^0W$ , which can also be thought of (using the evident action of  $\Gamma$  on  $E^0W = [W, E]$ ) as the map  $u \mapsto \gamma^{-1}.u$ . The set  $E^0W$  is finite by an easy thick subcategory argument [12, Theorem 8.5] so the stabiliser of  $u$  has finite index in  $\Gamma$ . As the subgroup  $1 + pD < \Gamma$  is a finitely topologically generated pro- $p$  group of finite index in  $\Gamma$ , we see from [4, Theorem 1.17] that every finite index subgroup of  $\Gamma$  is open. It now follows easily that the map  $u^*: \Gamma \rightarrow [W, E]$  is continuous when  $[W, E]$  is given the discrete topology. It follows in turn that if we give  $\Gamma$  the profinite topology and  $E^0E$  the natural topology then the inclusion map is continuous. We also know from [12, Proposition 11.5] that  $E^0E$  is Hausdorff. A continuous bijection from a compact space to a Hausdorff space is always a homeomorphism, and the lemma follows.  $\square$

**Theorem 12.** *There is a natural isomorphism  $E_0^\vee E = C(\Gamma, E_0)$ .*

*Proof.* Define a map  $\phi: \Gamma \times E_0^\vee E \rightarrow E_0$  by

$$\psi(\gamma, a) = (S \xrightarrow{a} E \hat{\wedge} E \xrightarrow{1 \hat{\wedge} \gamma} E \hat{\wedge} E \xrightarrow{\text{mult}} E).$$

Using [12, Propositions 11.1 and 11.3] we see that  $\psi$  is continuous, so we have an adjoint map  $\psi^\#: E_0^\vee E \rightarrow C(\Gamma, E_0)$ . We claim that this is an isomorphism. As both source and target are pro-free, it suffices to show that the reduction of  $\psi^\#$  modulo  $I_n$  is an isomorphism. Let  $K$  be the representing spectrum for the functor  $X \mapsto E^*/I_n \otimes_{K(n)^*} K(n)^*X$ ; this is a wedged of finitely many suspended copies of  $K(n)$ , and it can be made into an  $E$ -algebra spectrum with  $K_* = E_*/I_n$ . It is not hard to see that

$$(E_0^\vee E)/I_n = (E/I_n)_0 E = K_0 E,$$

and  $C(\Gamma, E_0)/I_n = C(\Gamma, E_0/I_n) = C(\Gamma, \mathbb{F}_q)$ . To analyse  $K_0 E$ , let  $x$  be the standard  $p$ -typical coordinate on  $\mathbb{G}$  and let  $F$  be the resulting formal group law over  $E_0$  — see [22, Appendix 2] for details and useful formulae. A standard Landweber exactness argument shows that  $K_0 E$  is the universal example of a ring  $R$  equipped with maps  $\mathbb{F}_q \xrightarrow{\alpha} R \xleftarrow{\beta} E_0$  and an isomorphism  $f: \beta_* F \rightarrow \alpha_* F$  of formal group laws. As  $\alpha_* F$  has height  $n$  we see that the same must be true of  $\beta_* F$ , so  $\beta(I_n) = 0$ , so we can regard  $\beta$  as a map  $\mathbb{F}_q = E_0/I_n \rightarrow R$ . The coefficients of  $F$  modulo  $I_n$  actually lie in  $\mathbb{F}_p \leq \mathbb{F}_q$  and there is only one map  $\mathbb{F}_p \rightarrow R$  so  $\alpha_* F = \beta_* F$ ; we just write  $F$  for this formal group law. Using the standard form for isomorphisms of  $p$ -typical FGL’s we can write  $f(x) = \sum_{k \geq 0}^F t_k x^{p^k}$ , where  $t_0$  is invertible because  $f$  is an isomorphism. (Readers may be more familiar with the graded case where one gets strict isomorphisms with  $t_0 = 1$ , but we are working with the degree zero part of two-periodic theories and  $t_0$  need not be 1 in this context.) As  $f$  commutes with  $[p]_F(x) = x^q$  we have  $\sum_k^F t_k x^{p^{n+k}} = \sum_k^F t_k^q x^{p^{n+k}}$  and thus  $t_k^q = t_k$ . In fact, this condition is sufficient for  $f$  to be a homomorphism of FGL’s (see [22, Appendix 2], for example) and we deduce that

$$R = (\mathbb{F}_q \otimes \mathbb{F}_q)[t_k \mid k \geq 0]/(t_0^{q-1} - 1, t_k^q - t_k) = C(D^\times, \mathbb{F}_q \otimes \mathbb{F}_q).$$

We next claim that  $\mathbb{F}_q \otimes \mathbb{F}_q$  can be identified with the ring  $F(C, \mathbb{F}_q)$  of functions from the Galois group  $C = \langle \phi \mid \phi^n = 1 \rangle$  to  $\mathbb{F}_q$ . Indeed, we can define a map  $\chi: \mathbb{F}_q \otimes \mathbb{F}_q \rightarrow F(C, \mathbb{F}_q)$  by  $\chi(a \otimes b)(\sigma) = a\sigma(b)$ . The  $\mathbb{F}_q$ -linear dual of this is the evident map  $\mathbb{F}_q[C] \rightarrow \text{End}_{\mathbb{F}_p}(\mathbb{F}_q)$  which is injective by Dedekind’s lemma on the independence of automorphisms, and this bijective by dimension count. Thus  $\chi$  is an isomorphism, and we obtain an isomorphism  $R = C(D^\times \ltimes C, \mathbb{F}_q)$ . After some comparison of definitions we see that this is the same as the map  $\psi^\#: K_0 E \rightarrow C(\Gamma, \mathbb{F}_q)$ , as required.  $\square$

**Definition 13.** We write  $J_k = E^{(k+1)} = E \hat{\wedge} \dots \hat{\wedge} E$  (with  $(k+1)$  factors). We can use the ring structure on  $E$  to assemble these objects into a cosimplicial spectrum and thus a cochain complex of spectra. We write  $C_\Gamma(\Gamma^{k+1}, E^0 X)$  for the set of continuous  $\Gamma$ -equivariant maps  $\Gamma^{k+1} \rightarrow E^0 X$  (where everything is topologised as in [12, Section 11]).

**Lemma 14.** *There is a natural isomorphism  $[X, J_k] = C_\Gamma(\Gamma^{k+1}, E^0 X)$ , which respects the evident cosimplicial structures.*

*Proof.* Given  $\underline{\gamma} = (\gamma_0, \dots, \gamma_k) \in \Gamma^{k+1}$ , we define

$$\nu(\underline{\gamma}) = (E^{(k+1)} \xrightarrow{\gamma_0 \wedge \dots \wedge \gamma_k} E^{(k+1)} \xrightarrow{\text{mult}} E).$$

We then define  $\phi_X : [X, J_k] \rightarrow \text{Map}(\Gamma^{k+1}, E^0 X)$  by  $\phi_X(a)(\underline{\gamma}) = \nu(\underline{\gamma}) \circ a : X \rightarrow E^0$ . Note that if  $\delta \in \Gamma$  then  $\nu(\delta \underline{\gamma}) = \delta \circ \nu(\underline{\gamma}) : J_k \rightarrow E$ ; this means that the map  $\phi_X(a) : \Gamma^{k+1} \rightarrow E^0 X$  is  $\Gamma$ -equivariant. The results of [12, Section 11] show that  $\phi_X(a)$  is continuous, so we can regard  $\phi_X$  as a map  $[X, J_k] \rightarrow C_\Gamma(\Gamma^{k+1}, E^0 X)$ .

Now let  $X$  be the dual of a generalised Moore spectrum [10, 12] of type  $S/I$  for some ideal  $I = (u_0^{a_0}, \dots, u_{n-1}^{a_{n-1}})$  so that  $E^0 X = E_0/I$ . We know from Theorem 12 and Lemma 10 that  $(E/I)_0 E = C(\Gamma, E_0/I)$  and that this is a free module over  $E_0/I$  so that

$$(E/I)_0 E^{(k)} = C(\Gamma, E_0/I)^{\otimes k} = C(\Gamma^k, E_0/I) = C_\Gamma(\Gamma^{k+1}, E_0/I).$$

After some comparison of definitions, we find that  $\phi_X$  is an isomorphism when  $X = D(S/I)$ . Moreover, the construction  $M \mapsto C(\Gamma^k, M)$  gives an exact functor from finite discrete Abelian groups to Abelian groups, so the construction  $X \mapsto C(\Gamma^k, E^0 X)$  gives a cohomology theory on the category of small  $K(n)$ -local spectra. By a thick subcategory argument, we deduce that  $\phi_X$  is an isomorphism when  $X$  is small. Now let  $X$  be a general  $K(n)$ -local spectrum, and let  $\{X_\alpha\}$  be the diagram of small spectra over  $X$ . As  $J_k$  is a pro-free  $E$ -module we see from Corollary 8 that  $[X, J_k] = \varprojlim_\alpha [X_\alpha, J_k]$ . We also see that  $C_\Gamma(\Gamma^{k+1}, E^0 X) = \varprojlim_\alpha C_\Gamma(\Gamma^{k+1}, E^0 X_\alpha)$ , and it follows that  $\phi_X$  is an isomorphism as claimed.  $\square$

**Corollary 15.** *There is a strongly convergent spectral sequence*

$$E_1^{st} = [X, J_s]^t = C_\Gamma(\Gamma^{s+1}, E^t X) \implies [X, \widehat{S}]^{t+s}.$$

*Proof.* The axiomatic treatment of Adams resolutions discussed in [18] can be transferred to many other triangulated categories; this will certainly work for unital stable homotopy categories in the sense of [11], and thus for  $\mathcal{K}$ . The spectra  $J_s$  clearly form an  $E$ -Adams resolution of  $\widehat{S}$ , so we have a spectral sequence whose  $E_1$  page is as described. Let  $i : \overline{E} \rightarrow \widehat{S}$  be the fibre of the unit map  $\eta : \widehat{S} \rightarrow E$ ; it is known that our spectral sequence is associated to the filtration of  $\widehat{S}$  by the spectra  $Y_s := \overline{E}^{(s)}$ . The map  $\eta \wedge 1 : E \rightarrow E \wedge E$  is split by the product map  $\mu : E \wedge E \rightarrow E$ , so  $i \wedge 1 = 0 : \overline{E} \wedge E \rightarrow E$ . It follows that if  $Z$  lies in the thick subcategory generated by  $E$  then the map  $i^{(s)} \wedge 1 : Y_s \wedge Z \rightarrow Z$  is zero for  $s \gg 0$ . However, we know from [12, Theorem 8.9] that  $\widehat{S}$  lies in this thick subcategory, so  $i^{(s)} = 0$  for  $s \gg 0$ . Using this and the definition of our spectral sequence, we see that it converges strongly to  $[X, \widehat{S}]^*$ .  $\square$

**Proposition 16.** *We have  $[E, S^{n^2-t}] = E_t$  as  $E_0$ -modules, and thus  $DE = \Sigma^{-n^2} E$  as  $E$ -module spectra. Moreover, this equivalence respects the evident actions of  $\Gamma$ .*

*Proof.* We use the spectral sequence of Corollary 15. To analyse the  $E_1$  term, we claim that  $E^0 E = \mathbb{Z}_p[[\Gamma]] \widehat{\otimes}_{\mathbb{Z}_p} E_0$  as left  $\Gamma$ -modules, where we let  $\Gamma$  act in the obvious way on  $\mathbb{Z}_p[[\Gamma]]$  and trivially on  $E_0$ . To see this, define maps  $\phi : E_0 \otimes \mathbb{Z}_p[[\Gamma]] \rightarrow E^0 E$  and  $\psi : \mathbb{Z}_p[[\Gamma]] \otimes E_0 \rightarrow E^0 E$  by

$$\phi(a \otimes [\gamma]) = (E \xrightarrow{\gamma} E \xrightarrow{\times a} E)$$

$$\psi([\gamma] \otimes a) = (E \xrightarrow{\times a} E \xrightarrow{\gamma} E).$$

Clearly  $\gamma' \cdot \phi(a \otimes [\gamma]) = \phi(\gamma'(a) \otimes [\gamma' \gamma])$  and  $\gamma' \cdot \psi([\gamma] \otimes a) = \psi([\gamma' \gamma] \otimes a)$  and  $\psi([\gamma] \otimes a) = \phi(\gamma(a) \otimes [\gamma])$  (because  $\gamma$  is a ring map). By dualising Theorem 12 we see that  $\phi$  extends to give an isomorphism  $E_0 \widehat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[\Gamma]] \rightarrow E^0 E$ , and it follows from the above formulae that  $\psi$  extends to give an isomorphism  $\mathbb{Z}_p[[\Gamma]] \widehat{\otimes}_{\mathbb{Z}_p} E_0 \rightarrow E^0 E$  with the required equivariance. Next, using the obvious description of  $E_0 = W[u_1, \dots, u_{n-1}]$  in terms of monomials, we see that  $E_0$  is isomorphic as a topological  $\mathbb{Z}_p$ -module to a product of copies of  $\mathbb{Z}_p$ , say  $E_0 = \prod_\alpha \mathbb{Z}_p$ . It follows that  $E^0 E = \prod_\alpha \mathbb{Z}_p[[\Gamma]]$  as  $\Gamma$ -modules, and using this that

$$C_\Gamma(\Gamma^{k+1}, E^0 E) = \prod_\alpha C_\Gamma(\Gamma^{k+1}, \mathbb{Z}_p[[\Gamma]]) = C_\Gamma(\Gamma^{k+1}, \mathbb{Z}_p[[\Gamma]]) \widehat{\otimes}_{\mathbb{Z}_p} E_0.$$

These identifications can easily be transferred to nonzero degrees, and they respect the cosimplicial structure, so it now follows from Proposition 5 that in our spectral sequence we have

$$E_2^{s,t} = \begin{cases} E^t & \text{if } s = n^2 \\ 0 & \text{otherwise.} \end{cases}$$

As the spectral sequence is strongly convergent, we have  $[E, S]^r = E^{r-n^2} = E_{n^2-r}$  as claimed. It is not hard to check that this is an isomorphism of  $E_0$ -modules, and it follows in the usual way that  $DE = \Sigma^{-n^2}E$  as  $E$ -module spectra. One can see from the construction that this is compatible with the action of  $\Gamma$ .  $\square$

**Proposition 17.** *There is a natural isomorphism  $E_t^\vee \hat{I} = \mathbb{I}(\pi_{n^2-t} ME)$ .*

*Proof.* First, we have

$$\begin{aligned} \mathbb{I}(ME_{t+n^2} \hat{I}) &= [\Sigma^{-n^2-t} E, \hat{I}^2 \hat{S}] \\ &= [\Sigma^{-n^2-t} E, \hat{S}] \\ &= E_{-t}. \end{aligned}$$

We can deduce from the above by a thick subcategory argument that  $K_* \hat{I}$  is finite in each degree and thus that  $\hat{I}$  is dualisable. (We could also have quoted this from [12]; the proof given there is only a slight perturbation of what we've just done.) This implies that  $\hat{I}X = F(X, \hat{I}) = DX \hat{\wedge} \hat{I}$  for all  $X \in \mathcal{K}$ . In particular, we have

$$E \hat{\wedge} \hat{I} = \Sigma^{n^2} DE \hat{\wedge} \hat{I} = \Sigma^{n^2} \hat{I}E,$$

so that  $E_t^\vee \hat{I} = \pi_{t-n^2} \hat{I}E = \mathbb{I}(\pi_{n^2-t} ME)$  as claimed.  $\square$

We next need to recall the calculation of  $\pi_* ME$  and its connection with local cohomology. As usual, we define  $E_*$ -modules  $E_*/I_k^\infty$  and  $u_k^{-1}E_*/I_k^\infty$  by the following recursive procedure: we start with  $E_*/I_0^\infty = E_*$ , we define  $u_k^{-1}E_*/I_k^\infty$  by inverting the action of  $u_k$  on  $E_*/I_k^\infty$ , and we define  $E_*/I_{k+1}^\infty$  to be the cokernel of the evident inclusion  $E_*/I_k^\infty \rightarrow u_k^{-1}E_*/I_k^\infty$ . It is also well-known that there is a unique way to make  $\Gamma$  act on all these modules that is compatible with the  $E_*$ -module structure and the short exact sequences

$$E_*/I_k^\infty \rightarrow u_k^{-1}E_*/I_k^\infty \rightarrow E_*/I_{k+1}^\infty.$$

(The point is that inverting  $u_k$  is the same as inverting  $v_k$ , and for each finitely generated submodule  $M_* < E_*/I_k^\infty$  there is some  $N$  such that  $v_k^{p^N}$  acts  $\Gamma$ -equivariantly on  $M_*$ ; this determines the action on  $u_k^{-1}E_*/I_k^\infty$ .)

One can define  $E$ -module spectra  $E/I_k^\infty$  and  $u_k^{-1}E/I_k^\infty$  and cofibrations

$$E/I_k^\infty \rightarrow u_k^{-1}E/I_k^\infty \rightarrow E/I_{k+1}^\infty$$

such that everything has the obvious effect in homotopy; this can either be done by Bousfield localisation [21] or by the theory of modules over highly structured ring spectra [5, Section 6]. The former approach has the advantage that it is manifestly  $\Gamma$ -equivariant. It is easy to see from the definitions that  $M(u_k^{-1}E/I_k^\infty) = 0$  so that  $M(E/I_k^\infty) = \Sigma^{-1}M(E/I_{k+1}^\infty)$ , and also that  $M(E/I_n^\infty) = E/I_n^\infty$ ; it follows that  $ME = \Sigma^{-n^2}E/I_n^\infty$ . All this is well-known and we record it merely to fix the grading conventions.

We next consider some parallel facts in local cohomology. Recall that if  $R$  is a Noetherian local ring with maximal ideal  $\mathfrak{m}$  and  $M$  is an  $R$ -module then the local cohomology groups  $H_{\mathfrak{m}}^i(M)$  may be defined as  $\lim_{\rightarrow k} \text{Ext}_R^i(R/\mathfrak{m}^k, M)$ ; see [1] (for example) for an account of these groups. Clearly  $\text{Aut}(R)$  acts naturally on  $H_{\mathfrak{m}}^*(R)$ , and if any element  $u \in \mathfrak{m}$  acts invertibly on  $M$  then  $H_{\mathfrak{m}}^i(M) = 0$ . In particular, the element  $u_k \in I_n$  acts invertibly on  $u_k^{-1}E_*/I_k^\infty$ , so  $H_{I_n}^*(u_k^{-1}E_*/I_k^\infty) = 0$ . Thus, the short exact sequence

$$E_*/I_k^\infty \rightarrow u_k^{-1}E_*/I_k^\infty \rightarrow E_*/I_{k+1}^\infty$$

gives rise to a  $\Gamma$ -equivariant isomorphism

$$H_{I_n}^s(E_*/I_k^\infty) \simeq H_{I_n}^{s-1}(E_*/I_{k+1}^\infty).$$

It is also standard that  $H_{I_n}^0(E_*/I_n^\infty) = E_*/I_n^\infty$  and the other local cohomology groups of this module vanish. We thus end up with a  $\Gamma$ -equivariant isomorphism

$$H_{I_n}^n E_* = E_*/I_n^\infty,$$

and the other local cohomology groups of  $E_*$  vanish. (This can be obtained more directly using an appropriate stable Koszul complex, but  $\Gamma$  does not act on that complex and the whole question of equivariance is rather subtle from that point of view.)

The next ingredient that we need is a certain “residue map”

$$\rho: E_0/I_n^\infty \otimes_{E_0} \Omega^{n-1} \rightarrow \mathbb{Q}_p/\mathbb{Z}_p.$$

Here  $\Omega^{n-1}$  is the top exterior power of the module of Kähler differentials for  $E_0$  relative to  $W$ ; it is freely generated over  $E_0$  by the element  $\epsilon := du_1 \wedge \dots \wedge du_{n-1}$ . The definition of  $\rho$  involves the map  $\tau: W \rightarrow \mathbb{Z}_p$  that sends  $a \in W$  to the trace of the map  $x \mapsto ax$ , considered as a  $\mathbb{Z}_p$ -linear endomorphism of  $W$ . This induces a map  $W/p^\infty \rightarrow \mathbb{Z}/p^\infty$  which we also call  $\tau$ . We define  $\tau': W \rightarrow \text{Hom}(W, \mathbb{Z}_p)$  by  $\tau'(a)(b) = \tau(ab)$ ; it is a standard fact of algebraic number theory that this is an isomorphism.

Given a multiindex  $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{Z}^{n-1}$  we define  $u^\alpha = \prod_i u_i^{\alpha_i}$ ; we also write  $\theta$  for the multiindex  $(1, \dots, 1)$ . The group  $E_0/I_n^\infty \otimes_{E_0} \Omega^{n-1}$  is a direct sum of copies of  $W[1/p]/W = W \otimes \mathbb{Q}_p/\mathbb{Z}_p$  indexed by monomials  $u^{-\theta-\alpha}\epsilon$  for which  $\alpha_i \geq 0$  for all  $i$ . We put

$$\rho\left(\sum_{\alpha} a_{\alpha} u^{-\theta-\alpha} \epsilon\right) = \tau(a_0).$$

**Proposition 18.** *The map  $\rho$  is  $\Gamma$ -invariant.*

*Proof.* This is an instance of the well-known principle of invariance of residues. Most of the formulae involved are very old, and were originally interpreted in a complex analytic context; starting in the 1960’s they were transferred into algebraic geometry [7, 16] but only in the very recent paper [13] do they appear in the particular technical context that we need. First note that  $W$  is the smallest closed subring of  $E_0$  containing all the  $(q-1)$ ’th roots of unity in  $E_0$ , so it is preserved by  $\text{Aut}(E_0)$  and in particular by  $\Gamma$ . Section 5 of the cited paper gives a canonical map

$$\text{res}: H_{I_n}^{n-1}(E_0/p^\infty \otimes_{E_0} \Omega^{n-1}) \rightarrow W/p^\infty,$$

and the short exact sequence  $E_0 \rightarrow E_0[1/p] \rightarrow E_0/p^\infty$  gives an isomorphism

$$E_0/I_n^\infty \otimes_{E_0} \Omega^{n-1} = H_{I_n}^n(\Omega^{n-1}) = H_{I_n}^{n-1}(E_0/p^\infty \otimes_{E_0} \Omega^{n-1}),$$

and  $\tau$  gives a canonical map  $W/p^\infty \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$ . By putting all these together, we get a map  $\rho': E_0/I_n^\infty \otimes_{E_0} \Omega^{n-1} \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$ . All the constructions involved are functorial and thus  $\Gamma$ -invariant. By examining the formulae in [13, Section 5] we see that  $\rho = \rho'$ .  $\square$

**Proposition 19.** *There is a natural isomorphism  $\mathbb{I}(\pi_t ME) = \Omega^{n-1} \otimes_{E_0} E_{-n-t}$ .*

*Proof.* This is essentially the local duality theorem (see [13, Theorem 5.9] for example). We first translate the claim using the isomorphism  $ME = \Sigma^{-n} E/I_n^\infty$ ; it now says that  $\mathbb{I}(E_t/I_n^\infty) = \Omega^{n-1} \otimes E_{-t}$ . Using the evident isomorphism  $E_{-t} = \text{Hom}_{E_0}(E_t, E_0)$  we can reduce to the case  $t = 0$ . In that case the construction above gives a  $\Gamma$ -invariant element  $\rho \in \mathbb{I}(E_0/I_n^\infty \otimes \Omega^{n-1})$  and thus a  $\Gamma$ -equivariant map  $\phi: \Omega^{n-1} \rightarrow \mathbb{I}(E_0/I_n^\infty)$ , defined by  $\phi(\lambda)(a) = \rho(a\lambda)$ . This satisfies

$$\phi\left(\sum_{\beta} b_{\beta} u^{\beta} \epsilon\right)\left(\sum_{\alpha} a_{\alpha} u^{-\theta-\alpha}\right) = \sum_{\alpha} \tau(a_{\alpha} b_{\alpha}).$$

As the map  $a \otimes b \mapsto \tau(ab)$  is a perfect pairing, it is easy to conclude that  $\phi$  is an isomorphism.  $\square$

We next recall the theorem of Hopkins and Gross, which identifies  $\Omega^{n-1}$  in terms of certain modules whose  $\Gamma$ -action is easier to understand. First, we write  $\omega = E_2$ , considered as an  $E_0$ -module with compatible  $\Gamma$ -action. To explain the notation, note that  $J := \tilde{E}^0 \mathbb{C}P^\infty$  is the group of formal functions on  $\mathbb{G}$  that vanish at zero. Thus  $J/J^2$  is the cotangent space to  $\mathbb{G}$  at zero, which is naturally isomorphic to the group of invariant one-forms on  $\mathbb{G}$ , which is conventionally denoted by  $\omega$ . On the other hand  $J/J^2$  is also naturally identified with  $\tilde{E}^0 \mathbb{C}P^1 = \tilde{E}^0 S^2 = E_2$ , so the notation is consistent.



**Theorem 20** (Gross-Hopkins). *There is a natural  $\Gamma$ -equivariant isomorphism  $\Omega^{n-1} \simeq \omega^{\otimes n}[\det]$ .*

*Proof.* See [8, Corollary 3] (which relies heavily on [6]).  $\square$

We can now give our proof of the main topological duality theorem.

*Proof of Theorem 2.* This follows from Propositions 17 and 19 and Theorem 20 after noting that  $\omega^{\otimes n} = E_{2n}$ .  $\square$

**Remark 21.** We conclude by comparing our formulation of the theorem with various other possibilities considered by other authors. Firstly, there are several alternative descriptions of the spectrum  $\hat{I}$ : we have

$$\hat{I} = IMS = L_{K(n)}IS = L_{K(n)}IL_{K(n)}S.$$

To see this, let  $X$  be a finite spectrum of type  $n$ , so that  $L_{K(n)}X = MX = L_nX$ , and smashing with  $X$  commutes with all the functors under consideration. The spectra listed above are all  $K(n)$ -local, and the canonical maps between them can be seen to become isomorphisms after smashing with  $X$ , and the claim follows.

Next, there are several alternatives to our spectrum  $E$ , most importantly the spectrum  $E'$  constructed using the Witt ring of  $\overline{\mathbb{F}}_p$  rather than  $\mathbb{F}_q$ . This is in some ways a more canonical choice, but it has the disadvantage that  $\pi_0 E'$  is not compact (although it is linearly compact) and  $\text{Aut}(E')$  is not a  $p$ -adic analytic group. In any case,  $E'$  is a pro-free  $E$ -module, so it is not hard to translate information between the two theories if desired.

Finally, some other authors use the functor  $\lim_{\leftarrow I} E'_0(S/I \wedge X)$  in place of  $E_0^\vee X$ . Hopkins writes this as  $\mathcal{K}(X)$ , and calls it the Morava module of  $X$ . One can show that

$$\mathcal{K}(X) = \pi_0(L_{K(n)}(E' \wedge X)) = E'_0 \hat{\otimes}_{E_0} E_0^\vee X$$

for large classes of spectra  $X$ , for example all  $K(n)$ -locally dualisable spectra, or spectra for which  $K(n)_*X$  is concentrated in even degrees. However, this equality can fail (for example when  $X = \bigvee_I S/I$ ) and the formal properties of the functor  $E_0^\vee X$  seem preferable in general.

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